Review for heat equation.

Physical meaning.

(Assume $x \in \Omega$, $\partial\Omega$ is C^1 ...)

Consider $u(t,x)$, which represents something like energy at space x, time t

And here, we can take $\int_\Omega u(t,x)\,dx$ as the whole energy.

$\Rightarrow \frac{d}{dt}\int_\Omega u(t,x)\,dx$ is the change of energy w.r.t time.

And let $J(t,x)$ be the energy flux.

$$\Rightarrow \frac{d}{dt}\int_\Omega u(t,x)\,dx = -\int_{\partial\Omega} J(t,x)\cdot n\,dS = -\int_{\partial\Omega} \nabla J\,dx$$

Sometimes $J(x,t) = -k\nabla u(t,x)$, $\Rightarrow$ [illegible]

Three kind of boundary condition

① $u|_{\partial\Omega} = \nu$ (Dirichlet)

② $\frac{\partial u}{\partial n}|_{\partial\Omega} = \nu$ (Newmann)

③ $-k\frac{\partial u}{\partial n}|_{\partial\Omega} = H(u-\nu)|_{\partial\Omega}$ (Mixed)

(Here, assume $u_t = a^2\Delta u$'s $a=1$, otherwise, $v(t,x) = u(t,ax)$).

I) Classical solution is unique:

$$\begin{cases} u_t = \Delta u + f \\ u(0,x) = \varphi(x) \\ u|_{\partial\Omega} = \text{bdry condition.} \end{cases} \Leftrightarrow \begin{cases} u_t = \Delta u \\ u(0,x) = 0 \\ u|_{\partial\Omega} = \text{homogeneous bdry condition} \end{cases}$$

only has classical solution 0

Proof: Consider $e(t) = \frac{1}{2}\int_U u^2(t,x)\,dx$, $e(0) = 0$

$$\Rightarrow \dot{e}(t) = \int_\Omega u(t,x)\,u_t(t,x)\,dx$$

$$= \int_\Omega u(t,x)\,\Delta u\,dx$$

$$= \int_{\partial\Omega} u\frac{\partial u}{\partial n}\,dS - \int_\Omega |\nabla u(t,x)|^2\,dx$$

≤ 0

II). To give a way to deduce a solution on $\mathbb{R}^d$.
It's natural to review some Fourier transform.

- $(f*g)^\wedge = \hat{f}\,\hat{g}$
- $\int \hat{f} g = \int f \hat{g}$

$\}$ First holds for $L^1(\mathbb{R}^d)$, other condition need some additional knowledge.

- If $f, \hat{f} \in L^1 \Rightarrow f = (\hat{f})^\vee$. (inversion formula).

- For $f \in C_c^\infty(\mathbb{R}^d)$, $(D^\alpha f)^\wedge = (2\pi i)^{|\alpha|} \xi^\alpha \hat{f}$
(Just integration by parts)

Now, $\begin{cases} u_t = \Delta u \\ u(0,x) = \varphi(x) \end{cases} \Rightarrow \begin{cases} \hat{u}_t = -4\pi^2 \xi^2 \hat{u} \\ \hat{u}(0,\xi) = \hat{\varphi}(\xi) \end{cases}$

$\Rightarrow \hat{u}(\xi) = e^{-4\pi^2\xi^2 t} \hat{\varphi}(\xi)$

So $(?)^\wedge = e^{-4\pi^2\xi^2 t}$
By residue formula, one can show that in $\mathbb{R}^d$

$$\left(\frac{1}{(\sqrt{4\pi t})^d} e^{-\frac{x^2}{4t}} \right)^\wedge = e^{-4\pi^2 \xi^2 t}$$

Set $G_t(x) = \frac{1}{(\sqrt{4\pi t})^d} e^{-\frac{x^2}{4t}} \Rightarrow u(t,x) \overset{\text{supposed to be}}{=} G_t(x) * \varphi(x)$

Thm: Consider on $\mathbb{R}^d$.

$\begin{cases} u_t = \Delta u, \ t > 0, x \in \mathbb{R}^d \\ u(0,x) = \varphi(x), \end{cases}$, $\phi \in C(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$

① $u(t,x) = G_t * \phi \in C^\infty((0,\infty) \times \mathbb{R}^d)$, $u_t = \Delta u$

② $\lim\limits_{(t,x)\to(0,x_0)} u(t,x) = \phi(x^0)$, $x^0 \in \mathbb{R}^d$.

Pf: Lemma: $f, g \in L^1(\mathbb{R}^d)$, $\partial_{x_i} f \in C(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$

Then $\partial_{x_i}(g*f) = g*\partial_{x_i} f$.

Lemma's pf: consider $\frac{1}{h}\int (f(x-y+he_i) - f(x-y))\, g(y)\, dy$

From DCT: $h \to 0$, dominated by $2|\partial_{x_i} f|_{L^\infty}\, g(y) \cdots \Rightarrow \square$.

So by this, we can. just transform U_t, $U_{x_ix_i}$ to G_t, $G_{x_ix_i}$

$(\partial_t(G*\varphi) = G_t*\varphi, \dots) \Rightarrow C^\infty$ and satisfies ①.

For the initial condition. consider (Approximation identity).

Lemma $k_n(x)$ non-negative cts, s.t

① $\int_{\mathbb{R}^d} |k_n(x)|\, dx = 1$ ② $\forall \varepsilon > 0$, $\lim_{n\to\infty} \int_{|x| \geq \varepsilon} k_n(x)\, dx = 0$.

Then $\forall g(x)$ is continous $\overset{\text{near } 0}{\vee}$ and. bdd .

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} k_n(x) g(x) = g(0) = \int g(x) \delta(x)\, dx$$

Lemma's pf: consider $\left|\int_{\mathbb{R}^d} k_n(x)(g(x) - g(0))\right|$

$$\leq \int_{\|x\| \leq \varepsilon} |k_n(x)|\, \underbrace{|g(x) - g(0)|}_{\downarrow} + \int_{\|x\| \geq \varepsilon} k_n(x)\, \underbrace{|g(x) - g(0)|}_{\downarrow} \dots$$

Leverage g is cts Leverage g is bdd.

From the Lemma, as $t \to 0$, we can take $G_t(x)$ as an approximation identity.

On $\mathbb{R}^d$, there exists non-homogeneous condition

$$\begin{cases} U_t = \Delta U + f \\ U|_{t=0} = 0. \end{cases} \qquad Ⓐ$$

Duhamel principle Let $f \in C^{1,2}([0,\infty)\times\mathbb{R}^d) \cap C_c([0,\infty)\times\mathbb{R}^d)$

$U(t,x) = \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y) f(s,y)\,dy\,ds$ solves Ⓐ

(初值依靠源项).

Ⅲ) On bounded domain, the story is different.

Before introducing further, let's consider the Strum-Liouville boundary problem:

$$\begin{cases} -\Delta U = \lambda U, \ \Omega. \\ \alpha U + \beta \frac{\partial U}{\partial n} = 0, \partial\Omega, \ \alpha,\beta \geq 0, \ \alpha+\beta>0 \end{cases} \Rightarrow (\lambda, U) \text{ is called an eigenpair.}$$

(Here, α, β can be changed to $\alpha(x), x\in\partial\Omega$ and $\beta(x), x\in\partial\Omega$)

We have:

① If $(\lambda_1, U), (\lambda_2, V)$ are eigenpairs, then we have:

$$\int_\Omega -\Delta U V\,dx = -\int_{\partial\Omega} \frac{\partial U}{\partial n} V\,dS + \int_\Omega \nabla U\cdot\nabla V\,dx$$

$$= -\int_{\partial\Omega} \frac{\partial V}{\partial n} U\,dS + \int_\Omega \nabla U\cdot\nabla V\,dx = \int_\Omega -U\Delta V\,dx$$

$$\begin{pmatrix} U & \frac{\partial U}{\partial n} \\ V & \frac{\partial V}{\partial n} \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0 \Rightarrow U\frac{\partial V}{\partial n} - \frac{\partial U}{\partial n}V = 0$$

② Each eigenvalue of Δ is real

③ Eigenvalues are countable, $0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 < \cdots$, $\lim_{n\to\infty} \lambda_n = \infty$

④ $\lambda_1 \neq \lambda_2$ are eigenvalues. $\lambda_1 \Leftrightarrow U_1$, $\lambda_2 \Leftrightarrow U_2 \Rightarrow \int_\Omega U_1 U_2 = 0$, i.e. they are orthogonal in $L^2(\Omega)$

In fact, the eigenfunction forms an orthogonal basis for $L^2(\Omega)$.

We only prove why eigenvalue $\lambda \geq 0$ and the orthogonality.

(i) Assume that (λ, u) is an eigenpair.

$$\lambda \int_\Omega u^2 \, dx = \int_\Omega -\Delta u \, u \, dx$$

$$= -\int_{\partial\Omega} \frac{\partial u}{\partial n} u \, \overset{\geq 0}{ds} + \int_\Omega |\nabla u|^2 \overset{\geq 0}{dx} \geq 0$$

(ii) (λ, u), (η, v) is an eigenpair.

Then. we have: $\lambda \int_\Omega uv = \int_\Omega -\Delta u) v \, dx$

$$= -\int_{\partial\Omega} \frac{\partial u}{\partial n} v \, ds + \int_\Omega \nabla u \cdot \nabla v$$

$$= -\int_{\partial\Omega} \frac{\partial v}{\partial n} u \, ds + \int_\Omega \nabla u \cdot \nabla v = \eta \int_\Omega uv \Rightarrow \square$$

Now, we propose a method to solve HE in 1 dimension.

$$\begin{cases} u_t = \Delta u + f \\ u(0,x) = \varphi(x) \quad \text{for } x \qquad x \in (0,l) \\ \alpha_1 u_x(0) + \beta_1 u(0) = g_1(t) \\ \alpha_2 u_x(l) + \beta_2 u(l) = g_2(t) \end{cases}$$

Naturally, we always focus $\begin{cases} u(t,0) = g_1(t) \\ u(t,l) = g_2(t) \end{cases}$ or $\begin{cases} u'(t,0) = g_1(t) \\ u'(t,l) = g_2(t) \end{cases}$ or mixed.

Use $\tilde{u}(t,x) = u(t,x) - \frac{l-x}{l} g_1(t) \begin{cases} -\frac{x}{l} g_2(t) \\ \end{cases}$
$\begin{cases} + \frac{(x-l)^2}{2l} g_1(t) \end{cases} \begin{cases} \\ -\frac{x^2}{2l} g_2(t) \end{cases} \Rightarrow$ deal. respectively

So, we just need to deal with $g_1(t) \equiv g_2(t) \equiv 0$

Consider $X_n(x)\, T_n(t)$ which satisfies $\begin{cases} U_t = \Delta U \\ \alpha_1 U_x(0) + \beta_1 U(0) = 0 \\ \alpha_2 U_x(l) + \beta_2 U(l) = 0 \end{cases}$

Then $\begin{cases} X_n(x)\, T_n'(t) = X_n''(x)\, T_n(t) \Rightarrow \dfrac{X_n''(x)}{X_n(x)} = \dfrac{T_n'(t)}{T_n(t)} = -\lambda_n \\ \alpha_1 X_n'(0) + \beta_1 X_n(0) = 0 \\ \alpha_2 X_n'(l) + \beta_2 X_n(l) = 0 \end{cases}$

From above, we have $\lambda_n \geq 0$ and for all eigenvalues λ, the eigenfunctions $\{X_n(x)\}$ forms a orthogonal basis in $L^2(0, l)$

Hence, if $\varphi(x) \in L^2(0, l) \Rightarrow$ we can decompose $\varphi(x) = \sum_{n=1}^{\infty} \varphi_n X_n(x)$
$\left(\varphi_n = \dfrac{\int_0^l \varphi(x) X_n(x)\, dx}{\int_0^l X_n^2(x)\, dx} \right)$.

if $f(x) \in L^2(0, l) \Rightarrow$ We can decompose $f(x) = \sum_{n=1}^{\infty} f_n(t) X_n(x)$

Back to the original situation.

We just need to let $\begin{cases} T_n'(t) = -\lambda T_n(t) + f_n(t) \\ T_n(0) = \varphi_n \end{cases}$, then $\sum_{n=1}^{\infty} T_n(t) X_n(x)$ solves the HE.

IV). Maximum principle and related energy estimate. comparison principle.

Maximum principle (HE version, and we will state a generalized version here). $\triangleq U_T$ For U is bounded. and open
If $u \in C^{1,2}(\,(0,T] \times U\,) \cap C(\overline{U_T})$ and consider the operator:

$$Lu = \sum_{i,j=1}^{\overset{\nearrow \dim U}{n}} a_{ij}(t,x)\,\overset{\nearrow \partial x_i \partial x_j}{\partial_{ij}} u \overset{+\sum_{i=1}^{n} b_i(t,x)\partial_i u}{\checkmark}$$, where $A = (a_{ij}(t,x))$ is positive semidefinite

If $u_t - Lu \le 0$, then $\max\limits_{\overline{U_T}} u = \max\limits_{\partial U_T} u$

Proof: First, assume $u_t - Lu < 0$.

If there exists $(t_0, x_0) \in [0,T] \times U$, s.t $u(t_0, x_0) = \max\limits_{\overline{U_T}} u$

$$\Rightarrow \begin{cases} u_t \ge 0, \ \nabla^2 u(t_0,x_0) \le 0 \Rightarrow A\nabla^2 u(t_0,x_0) \le 0 \\ \partial_i u = 0 \end{cases} \Bigg\} \ Lu \le 0$$

$\Rightarrow u_t - Lu \ge 0 \Rightarrow$ Contradiction.

If $u_t - Lu \le 0$, consider $v_\varepsilon(t,x) = u - \varepsilon t \le u$, let $\varepsilon \to 0^+$ is enough.

As a particular example, we have: if $u_t - \Delta u \le 0$ for $u \in C^{1,2}_{\cdot\cdot} \cap C$

Then $\max\limits_{\overline{U_T}} u = \max\limits_{\partial U_T} u$

From this, we can get the comparison principle

For $u, v \in C^{1,2}(U_T) \cap C(\overline{U_T})$. if

$$\begin{cases} u_t - Lu \le v_t - Lv \\ u|_{\partial U_T} \le v|_{\partial U_T} \end{cases} \Rightarrow u \le v \text{ in } \overline{U_T}.$$

Using this, we can give another proof of the uniqueness of heat equation.

Using this, it's easy to get:

L^∞-stability: Let U_i be the classical solution $C^{1,2}(U_T)$ $\cap C(\bar{U}_T)$, s.t $\begin{cases} \partial_t U_i = \Delta U_i + f_i, U_T \\ U_i|_{\partial\Omega} = g_i, \ U_i|_{t=0} = \varphi_i \end{cases}$

Then: $\max\limits_{\bar{U}_T} |U_1 - U_2| \le T\|f_1 - f_2\|_{L^\infty} + \|g_1 - g_2\|_{L^\infty} + \|\varphi_1 - \varphi_2\|_{L^\infty}$

Now, we propose L^∞-stability for mixed boundary condition. Similarly, we need a comparison principle.

- $u \in C^{1,2}(U_T) \cap C^{0,1}(\bar{U}_T)$ satisfy

$\begin{cases} Lu = u_t - \Delta u \geq 0 \text{ in } U_T \\ u|_{t=0} \geq 0 \text{ in } U \\ \frac{\partial u}{\partial n} + \beta u|_{\partial U} \geq 0, t \in [0,T] \end{cases}$ $U = (0,1)$, $\beta : \partial U \to [0,\infty)$. $\Rightarrow u \geq 0$ in $\bar{U}_T$

Proof: Since $Lu = u_t - \Delta u \geq 0$, we have $\min\limits_{\bar{U}_T} u = \min\limits_{\partial U_T} u$

If $\exists (t^*, x^*) \in \partial U_T$, s.t $u(t^*, x^*) \le 0$

Notice that $\nabla u(t^*, x^*) \cdot n \le 0$, since (t^*, x^*) is minimal point.

So, $\frac{\partial u}{\partial n} + \beta u|_{\partial U} \le 0$, if $\frac{\partial u}{\partial n} + \beta u|_{\partial U} > 0 \Rightarrow$ Contradiction

If not, consider $w(t,x) = 2t + (x - \frac{1}{2})^2$, $\begin{cases} Lw = 0, w|_{t=0} \geq 0 \text{ in } U \\ \frac{\partial w}{\partial n} + \beta u|_{\partial U} \geq C > 0. \end{cases}$

And consider $u_\varepsilon = u + \varepsilon w \rightrightarrows u$ as $\varepsilon \to 0^+$. ∎

Hence, we get: If $u \in C^{1,2}(U_T) \cap C^{0,1}(\bar{U}_T)$, s.t.

$\begin{cases} Lu = u_t - \Delta u = f(t,x) \\ u|_{t=0} = \Phi \\ \frac{\partial u}{\partial n} + \beta u|_{\partial U} = g(t,x) \end{cases} \Rightarrow \max\limits_{\bar{U}_T} u \le C(T+1)(\overset{\triangleq F}{\|f\|_{L^\infty}} + \overset{\triangleq \Phi}{\|\varphi\|_{L^\infty}} + \overset{\triangleq G}{\|g\|_{L^\infty}})$

Proof: Consider $Ft+\underline{\Phi}+\frac{a}{c}w(t,x)\cdot \pm u(t,x)$. $\boxtimes$

Here, we give an application of maximum principle to finish the discussion about HE.

- For HE on the whole space, if $u\in C^{1,2}([0,T]\times\mathbb{R}^d)\cap C([0,T]\times\mathbb{R}^d)$ solves $\begin{cases}\partial t=\Delta u\\ u|_{t=0}=\varphi\end{cases}$ with $|u(t,x)|\leq Ae^{a|x|^2}$ for some $A>0$, $a>0$.

Then $\max\limits_{[0,T]\times\mathbb{R}^d}|u|=\sup\limits_{\mathbb{R}^d}|\varphi|$.

Proof: Notice that: $\frac{\mu}{(T+\varepsilon-t)^{\frac{d}{2}}}e^{\frac{|x|^2}{4(T+\varepsilon-t)}}$ is a solution of $u_t=\Delta u$

$$(u_t=-\frac{d}{2}\frac{\mu\cdot(-1)}{(T+\varepsilon-t)^{\frac{d}{2}+1}}e^{\frac{|x|^2}{4(T+\varepsilon-t)}}+\frac{\mu}{(T+\varepsilon-t)^{\frac{d}{2}}}e^{\frac{|x|^2}{4(T+\varepsilon-t)}}\frac{-|x|^2\cdot(-1)}{4(T+\varepsilon-t)^2}$$

$$u_{x_i}=\frac{\mu}{(T+\varepsilon-t)^{\frac{d}{2}}}e^{\frac{|x|^2}{4(T+\varepsilon-t)}}\cdot\frac{2x_i}{4(T+\varepsilon-t)}=\frac{\mu x_i}{2(T+\varepsilon-t)^{\frac{d}{2}+1}}e^{\frac{|x|^2}{4(T+\varepsilon-t)}}$$

$$u_{x_ix_i}=\frac{\mu}{2(T+\varepsilon-t)^{\frac{d}{2}+1}}e^{\frac{|x|^2}{4(T+\varepsilon-t)}}+\frac{\mu x_i^2}{4(T+\varepsilon-t)^{\frac{d}{2}+2}}e^{\sim}\qquad)$$

So, consider $v(t,x)=u(t,x)-\frac{\mu}{(T+\varepsilon-t)^{\frac{d}{2}}}e^{\frac{|x|^2}{4(T+\varepsilon-t)}}$

Take $T+\varepsilon<\frac{1}{4a}$, take L big enough, consider $[0,T]\times B_L(0)$

Then, we have:

$$\partial_t v=\Delta v\Rightarrow\max_{[0,T]\times B_L(0)}v=\max_{\partial(\sim)}v.$$

When $t=0$, $v(0,x)\leq\varphi(x)$ On $\partial B_L(0)$, $v(t,x)\leq Ae^{a|L|^2}-\frac{\mu}{(T+\varepsilon)^{\frac{d}{2}}}e^{\frac{|x|^2}{4(T+\varepsilon)}}$

$$\leq Ae^{a|L|^2}-\frac{\mu}{(T+\varepsilon)^{\frac{d}{2}}}e^{\frac{|L|^2}{4(T+\varepsilon)}}\leq 0\leq\sup_{\sim}|\varphi|$$

$\Rightarrow \sup\limits_{[0,T]\times B_L(0)} v(t,x) \leq \sup\limits_{\mathbb{R}^d} \varphi$. $\mu \to 0 \Rightarrow \sup\limits_{[0,T]\times B_L(0)} u(t,x) \leq \sup\limits_{\mathbb{R}^d} \varphi$.

Similarly, let $\tilde{v}(t,x) := -u(t,x) + \frac{\mu}{(T+\varepsilon-t)^{\frac{d}{2}}} e^{\frac{|x|^2}{4(T+\varepsilon-t)}}$'s minimum ...

$\Rightarrow \min\limits_{[0,T]\times B_L(0)} \tilde{v} \geq \min\limits_{\mathbb{R}^d} \varphi \Rightarrow \underset{\sim}{\min}\, u \geq \underset{\sim}{\min}\, \varphi$, or inf, just leverage bdd + closed $\Rightarrow$ cpt! ∎

Finally, we give the energy estimate

Let $u \in C^{1,2}(U_T) \cap C(\overline{U_T})$ solves $\begin{cases} (\partial_t - \Delta) u = f, & U_T \\ u|_{t=0} = \varphi \\ u|_{\partial U} = 0 \end{cases}$

Then, $\exists\, C = C(T)$, s.t

$$\sup_{0\leq t\leq T} \|u(t,\cdot)\|^2_{L^2(U)} + 2\int_0^T \|\nabla u(t,\cdot)\|^2_{L^2(U)}\, dt \leq C\left(\|\varphi\|^2_{L^2(U)} + \int_0^T \|f(t,\cdot)\|^2_{L^2(U)}\, dt\right)$$

Proof: : $\int_U u(u_t - \Delta u) = \int_U uf$, fix t.

Consider LHS $= \frac{1}{2}\frac{d}{dt}\int_U u^2(t,x)\,dx + \int_U |\nabla u(t,x)|^2 dx$

RHS $\leq \frac{1}{2}\int_U u^2 + f^2\, dx$

Let $e(t) = \int_U u^2(t,x)\,dx$, then $\dot{e}(t) \leq e(t) + 2\int_U f^2 dx$, $e(0) = \int_U \varphi^2(x)\,dx$

$\Rightarrow (e^{-t} e(t))' \leq e^{-t} \cdot 2\int_U f^2(t,x)\,dx \leq 2\int_U f^2(t,x)\,dx$

$\Rightarrow e^{-t} e(t) - \int_U \varphi^2(x)\,dx \leq 2\int_0^T\int_U f^2(t,x)\,dx\,dt$

Using this, the energy estimate is obvious. ∎